

A Probabilistic Approach for Gradient Inequalities on Time-Inhomogeneous Manifolds

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Abstract

Gradient inequalities of Hamilton type and Li-Yau type for positive solutions of the heat equation are established from a probabilistic viewpoint, which simplifies the proofs of the results of Bailesteanu, Cao and Pulemotov [4], Liu [14], or Sun [16].

Keywords: Ricci flow, heat equation, Li-Yau type inequality, g_t -Brownian motion
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1 Introduction and Main Result

Let M be a d -dimensional differential manifold without boundary which carries a $C^{1,\infty}$ -family of time-dependent Riemannian metrics $(g_t)_{t \in [0, T]}$, $T < \infty$. Let ∇^t, Δ_t be the Levi-Civita connection and the Laplace-Beltrami operator associated with the metric g_t respectively. For simplicity, we take the notations: for $X, Y \in TM$,

$$\mathcal{R}_t(X, Y) := \text{Ric}_t(X, Y) + \partial_t g_t(X, Y),$$

where Ric_t is the Ricci curvature tensor with respect to g_t . When $\mathcal{R}_t \equiv 0$, then $(g_t)_{t \in [0, T]}$ is a solution of a Ricci flow. Here and in what follows, the Ricci flow will mean (probabilistic convention):

$$\frac{\partial}{\partial t} g(x, t) = -\text{Ric}(x, t), \quad (x, t) \in M \times [0, T]. \quad (1.1)$$

Suppose a smooth positive function $u : M \times [0, T] \rightarrow \mathbb{R}$ satisfies the heat equation

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{2} \Delta_t u(x, t), \quad (1.2)$$

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on $M \times [0, T]$. In this paper, we look at the Ricci flow (1.1) combined with the heat equation (1.2). The study of the system (1.1)-(1.2) arose from R. Hamilton [9]. The system was further pursued e.g. in [8]. A large amount of work was done to understand several problems that are similar to (1.1) -(1.2) in a way or another, see e.g. [12, 5, 6, 4] and the reference within.

Bailesteanu, Cao, Pulemotov [4], Liu [14], and Sun[16] independently established a series of gradient estimate for positive solutions of the heat equation (1.2) on M under the Ricci flow by using some analysis methods. In this paper, we want to investigate them from a probabilistic viewpoint. When the metric is independent of t , this point of view has been work well for local estimates in positive harmonic function [2] and for Li-Yau type gradient estimates.

From technical viewpoint, the method is essentially due to [3]. The key to deal with these problems is how to construct suitable submartingales. Note that, compared with previous work, we simplify the proof and obtain the gradient inequalities more explicitly.

We will use the notations $|\cdot|, \langle \cdot, \cdot \rangle$ for the norm and metric with respect to g_t for simplicity. It is clear that they all depend on t . Denote $\|f\|_D$ stands for $\sup_D |f(x, t)|$. The following formulas for solutions of the heat equation on a manifold without boundary depend on the fact that the solutions are strictly positive.

When M is compact, the main result is presented as follows.

Theorem 1.1. *Suppose the manifold M is compact and $0 \leq \text{Ric}_t \leq k$, $t \in (0, T]$. Let $(M, g_t)_{t \in [0, T]}$ be a solution of the Ricci flow. Assume a smooth positive function $u : M \times [0, T] \rightarrow \mathbb{R}$ satisfies the heat equation (1.2). Then the following estimates*

$$\frac{|\nabla^t u|}{u} \leq \sqrt{\frac{2}{t} \log \frac{\|u\|_{M \times [0, T]}}{u}}; \quad \frac{|\nabla^t u|^2}{u^2} - \frac{\Delta_t u}{u} \leq kn + \frac{2n}{t} \quad (1.3)$$

hold for all $M \times (0, T]$.

Before moving on, let us give some comments on this theorem.

- (1) The first inequality in Theorem 1.1 does not need the curvature condition: $0 \leq \text{Ric}_t \leq k$, $t \in [0, T]$, which can be referred in [12], [6] by using some analysis methods.
- (2) In [4], the authors obtain $\frac{|\nabla^t u|^2}{u^2} - \frac{\Delta_t u}{u} \leq 2kn + \frac{n}{t}$ by using the maximal principle, which has a little difference from our result.
- (3) When the manifold M carries boundary, we turn to consider the system as follows: for $\lambda \geq 0$,

$$\begin{cases} \frac{\partial}{\partial t} g(x, t) = -\text{Ric}(x, t), & (x, t) \in M \times [0, T]; \\ \mathbb{I}_t = \lambda, & x \in \partial M. \end{cases} \quad (1.4)$$

Shen [15] give the proof of the short time exitance of the solution. See also [4] for more geometric explanation. Suppose a smooth positive function $u : M \times [0, T] \rightarrow \mathbb{R}$

satisfies the heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(x, t) = \frac{1}{2}\Delta_t u(x, t), & (t, x) \in [0, T] \times M; \\ N_t u(x, t) = 0, & x \in \partial M, \end{cases} \quad (1.5)$$

where N_t is the inward unit normal vector field of the boundary associated with g_t . It is easy to see that we can deal with the system (1.4)-(1.5) by using some reflecting g_t -Brownian motions. We will explain in Remark 2.3 that Theorem 1.1 also holds for the system (1.4)-(1.5).

When the manifold M is noncompact, we want to give the local version of the gradient estimates. Before of this, some pieces of notations should be introduced at this point. We write $\rho_t(x, y)$ for the distance between $x \in M$ and $y \in M$ with respect to g_t . Fix $x_0 \in M$ and $\rho > 0$. The notation $B_{\rho, T}$ stands for the set $\{(x, t) \in M \times [0, T] \mid \rho_t(x, x_0) < \rho\}$. We obtain the following results from Theorem 3.1 and Theorem 3.10 directly.

Theorem 1.2. *Suppose $(M, g_t)_{t \in [0, T]}$ is a complete solution of the Ricci flow (1.1). Assume that $|\text{Ric}_t| \leq k$ for some $k > 0$ and all $(x, t) \in B_{\rho, T}$, and a smooth positive function $u : M \times [0, T] \rightarrow \mathbb{R}$ satisfies the heat equation (1.2). Then for $\alpha > 1$,*

$$\frac{|\nabla^t u|^2}{u^2} - \alpha \frac{\Delta_t u}{u} \leq \frac{2\alpha^2 n}{t} + \frac{8\alpha^2 n \pi^2}{(4 - \pi)^2} \frac{n(1 + \alpha^4(\alpha - 1)^{-1}) + 3}{\rho^2} + \left(\left(\frac{4 + \pi}{4 - \pi} \right)^2 + \frac{\alpha}{\alpha - 1} \right) \alpha^2 k n,$$

and

$$\begin{aligned} \left| \frac{\nabla^t u}{u} \right|^2 &\leq 2 \left(\frac{1}{t} + \frac{4\pi^2(n + 7)}{(4 - \pi)^2 \rho^2} + \frac{16k\pi}{(4 - \pi)^2} \right) \left(4 + \log \frac{\|u\|_{B_{\rho, T}}}{u} \right), \\ \left| \frac{\nabla^t u}{u} \right|^2 - \frac{\Delta_t u}{u} &\leq \frac{2n}{t} + \frac{8n\pi^2(n + 3)}{(4 - \pi)^2 \rho^2} + \left(\frac{4 + \pi}{4 - \pi} \right)^2 n k + \left(\frac{32n\pi}{(4 - \pi)^2 \rho} + \sqrt{2nk} \right) \left\| \frac{|\nabla \cdot u|}{u} \right\|_{B_{\rho, T}} \end{aligned}$$

hold for all $(x, t) \in B_{\rho, T}$ with $t \neq 0$.

We know that in [4, 14, 16], the first two inequalities in Theorem 1.2 have been established. Although there are some difference on the explicit coefficients on the right hand side of these inequalities, the main order about t and ρ is the same. Also, see [4] for the further application to the Li-Yau Harnack inequalities.

In the following content, we investigate the announced problems for general geometric flows directly, not restrict to the Ricci flow. We will study the gradient estimate of Hamilton type and Li-Yau type in the rest two sections respectively. Remark that except as noted, all the discussions below are based on the manifold without boundary.

2 Gradient estimates of Hamilton type

In this section, we explain how submartingales related to positive solutions of the heat equation can be turned into gradient estimates of Hamilton type, i.e. the space-only

gradient estimates. Before of this, let us introduce basic formulas for solutions of the heat equation and some useful (sub)martingales, which are important throughout all the content.

Lemma 2.1. *Let $u = u(x, t)$ be a positive solution of (1.2) on $M \times [0, T]$. Then the following equations hold:*

$$\left(\frac{1}{2}\Delta_t - \partial_t\right)(u \log u) = \frac{1}{2} \frac{|\nabla^t u|^2}{u}; \quad (2.1)$$

$$\left(\frac{1}{2}\Delta_t - \partial_t\right) \frac{|\nabla^t u|^2}{u} = \frac{1}{u} \left| \text{Hess}_u^t - \frac{\nabla^t u \otimes \nabla^t u}{u} \right|^2 + \frac{\mathcal{R}_t(\nabla^t u, \nabla^t u)}{u}. \quad (2.2)$$

The two equalities in Lemma 2.1 can be checked directly. Equation (2.2) in Lemma 2.1 gives raise to some inequalities frequently used in the sequel and crucial for our approach. Let $f = \log u$. One can easily see that

$$\frac{1}{u} \left| \text{Hess}_u^t - \frac{\nabla^t u \otimes \nabla^t u}{u} \right|^2 = u |\text{Hess}_f^t|^2 \geq \frac{u}{n} (\Delta_t f)^2 = \frac{1}{nu} \left(\Delta_t u - \frac{|\nabla^t u|^2}{u} \right)^2. \quad (2.3)$$

Then if $\mathcal{R}_t \geq -k(t)$ on M for some $k \in C([0, T])$, then we have

$$\begin{aligned} \left(\frac{1}{2}\Delta_t - \partial_t\right) \frac{|\nabla^t u|^2}{u} &\geq \frac{1}{nu} \left(\Delta_t u - \frac{|\nabla^t u|^2}{u} \right)^2 + \frac{\mathcal{R}_t(\nabla^t u, \nabla^t u)}{u} \\ &\geq \frac{1}{nu} \left(\Delta_t u - \frac{|\nabla^t u|^2}{u} \right)^2 - k(t) \frac{|\nabla^t u|^2}{u} \\ &\geq -k(t) \frac{|\nabla^t u|^2}{u}. \end{aligned} \quad (2.4)$$

Let (X_t^T) be a $g(T-t)$ -Brownian motion on M (see [1] for the detailed construction), and $\{P_{s,t}\}_{0 \leq s \leq t \leq T}$ be the associated semigroup. Then it is easy to see that $P_{T-t,T}f$ is a solution of the equation (1.2). Based on Lemma 2.1, the following (sub)martingale can be constructed as in [3, Lemma 2.4] directly. Here, we skip the proof.

Lemma 2.2. *Let $u(x, t) = P_{T-t,T}f(x)$ be a positive solution of the heat equation (1.2) on $M \times [0, T]$. If $\mathcal{R}_t \geq -k(t)$ for some $k \in C([0, T])$, then for any $g(T-t)$ -Brownian motion (X_t^T) on M , the process*

$$H_t := h(t) \frac{|\nabla^t P_{T-t,T}f|^2}{P_{T-t,T}f} (X_{T-t}^T) + (P_{T-t,T}f \log P_{T-t,T}f)(X_{T-t}^T), \quad (2.5)$$

where $h(t) = \frac{1}{2} \int_0^t e^{-\int_s^t k(r) dr} ds$ is a local supermartingale (up to lifetime).

Remark 2.3. *Consider the system (1.4)-(1.5) for the manifold with boundary, (X_t^T) is a reflecting $g(T-t)$ -Brownian motion. We claim that H_t constructed in Lemma 2.2 is still a local submartingale. Since $\mathbb{I}_t = \lambda \geq 0$ and $N_t P_{T-t,T}f = 0$ by [7, Theorem 2.1(2)], it is easy to check that $N_t H_t \geq 0$, which leads to the assertion above. So the following discussion in this section also holds for the system (1.4)-(1.5).*

By Lemma 2.2, we have the following gradient estimate directly.

Theorem 2.4. *Let $u(x, t) = P_{T-t}f(x)$ be a positive solution of the heat equation (1.2) on a compact manifold M . Assume that $\mathcal{R}_t \geq -k(t)$ for $t \in (0, T]$. Then for $t \in (0, T]$, we have*

$$\left| \frac{\nabla^t u}{u} \right|^2(x, t) \leq \frac{2}{\int_0^t e^{-\int_s^t k(r) dr} ds} P_{T-t, T} \left(\frac{f}{P_{T-t, T}f} \log \frac{f}{P_{T-t, T}f} \right)(x).$$

Proof. When M is compact, the local submartingale H_t defined in (2.5) is a true submartingale. Then the following argument is the same as in [3, Theorem 3.1]. \square

By normalized f as $f^* = f/P_{T-t}f$ in Theorem 2.4, we conclude

Corollary 2.5 (Gradient inequality of Hamilton type). *Let M be a compact manifold such that $\mathcal{R}_t \geq -k(t)$ for some $k \in C([0, T])$. Suppose that $u(x, t)$ is a positive solution of the heat equation (1.2) on $M \times [0, T]$. Then for $t \in (0, T]$, we have*

$$\frac{|\nabla^t u|^2}{u^2}(x, t) \leq \frac{2}{\int_0^t e^{-\int_s^t k(r) dr} ds} \log \frac{\|u\|_{M \times [0, T]}}{u}. \quad (2.6)$$

In particular, if $\mathcal{R}_t \geq 0$, then for $t \in (0, T]$,

$$\frac{|\nabla^t u|}{u}(x, t) \leq \frac{1}{t^{1/2}} \sqrt{2 \log \frac{\|u\|_{M \times [0, T]}}{u(x, t)}}.$$

Our next step is to localize the arguments to cover the solution of the heat equation on $D \subset M \times [0, T]$, where D is a relatively compact open subset of $M \times (0, T]$ with nonempty smooth boundary. Our argument is based on the following assumption.

Assumption (A) $\mathcal{R}_t \geq -k_1$, $-k_2 \leq \partial_t g_t \leq k_3$ hold for some positive constants k_1, k_2, k_3 on the domain D .

Let $\varphi \in C^{1,2}(\bar{D})$ with $\varphi > 0$ and $\varphi|_{\partial D} = 0$. We denote

$$u_t = u(t, \cdot)(X_{T-t}^T), \quad \nabla^t u_t = \nabla^t u(t, \cdot)(X_{T-t}^T)$$

and $\Delta_t u_t = \Delta_t u(t, \cdot)(X_{T-t}^T)$. And write

$$q(t, x) = \frac{|\nabla^t u|^2}{u}(t, x) \quad \text{and} \quad q_t = q(t, X_{T-t}^T).$$

We now study the following process on D ,

$$S_t = \frac{|\nabla^t u_t|^2}{2u_t} - u_t(1 - \log u_t)^2 Z_t,$$

where

$$Z_t := \frac{C_1}{t} + \frac{C_2}{\varphi_t^2(X_{T-t}^T)} + C_3$$

for some constants $C_1, C_2, C_3 > 0$, which will be specified later. The following conclusion is derived by using the property of the process S_t .

Theorem 2.6 (Local gradient inequality of Hamilton type). *Assume (A) holds. Let u be a solution of the heat equation on D , which is positive and continuous on \bar{D} . Then*

$$\left| \frac{\nabla^t u}{u} \right|^2 (x, t) \leq 2 \left(\frac{1}{t} + \frac{\sup_D \{7|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\}}{\varphi_t^2(x)} + k_1 \right) \left(4 + \log \frac{\|u\|_{\bar{D}}}{u} \right)^2 \quad (2.7)$$

holds for $(x, t) \in D, t \neq 0$.

Proof. The proof is similar to the proof of [3, Theorem 6.1]. As the completeness of our exposition, we present it in the Appendix. \square

Now we choose $D = B_{\rho, T}$ for some $\rho > 0$. To specify the constants of (2.7) in terms of φ , an explicit choice for φ has to be done.

We fix $(x, t) \in B_{\rho/2, T}$. For any $y \in B_{\rho, T}$, $\rho_t(x_0, y) \leq \rho$ for $t \in [0, T]$. Consider on $B_{\rho, T}$,

$$\varphi(t, y) := \cos \frac{\pi \rho_t(x_0, y)}{2\rho}. \quad (2.8)$$

φ is nonnegative and bounded by 1, and φ vanishes on $\partial B_{\rho, T}$. Because $(x, t) \in B_{\rho/2, T}$, this gives us

$$\varphi_t(x) = \cos \frac{\pi \rho_t(x_0, x)}{2\rho} \geq 1 - \frac{\pi \rho_t(x_0, x)}{2\rho} \geq 1 - \frac{\pi}{4}. \quad (2.9)$$

By convention, $\Delta_t \varphi_t^{-2} = 0$ at points where φ_t^{-2} is not differentiable. As a consequence, all estimates in Theorem 2.6 remain valid with φ defined by (2.8).

We are now going to derive explicit expressions for the constants. To this end, we observe that

$$\|\nabla^t \varphi_t\|_{B_{\rho, T}} \leq \frac{\pi}{2\rho}. \quad (2.10)$$

Since $\mathcal{R}_t \geq -k_1$, $-k_2 \leq \partial_t g_t$ for $k_1, k_2 > 0$, by the index lemma, we have

$$\begin{aligned} -(\Delta_t - 2\partial_t)\varphi_t &= \sin \frac{\pi \rho_t(x_0, y)}{2\rho} \frac{\pi}{2\rho} (\Delta_t - 2\partial_t)\rho_t(x_0, \cdot)(y) + \cos \frac{\pi \rho_t(x_0, y)}{2\rho} \cdot \frac{\pi^2}{4\rho^2} |\nabla^t \rho_t|^2 \\ &\leq \sin \frac{\pi \rho_t(x_0, y)}{2\rho} \frac{\pi}{2\rho} \left[\frac{n-1}{\rho_t(x_0, y)} + (k_1 + k_2)\rho_t(x_0, y) \right] + \frac{\pi^2}{4\rho^2} \\ &\leq \frac{\pi^2 n}{4\rho^2} + \frac{\pi}{2}(k_1 + k_2). \end{aligned} \quad (2.11)$$

Then, we further have

$$\sup_{B_{\rho, T}} \{7|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\} \leq \frac{\pi^2(n+7)}{4\rho^2} + \frac{\pi}{2}(k_1 + k_2). \quad (2.12)$$

Especially, under assumption of Theorem 1.2, $k_1 = 0$, $k_2 = k_3 = k$. Then the second assertion of Theorem 1.2 can be derived by combining the estimates (2.9), (2.12) and Theorem 2.6.

3 Li-Yau type gradient inequalities

Our main task of this section is to extend the argument of the Li-Yau type inequalities to the inhomogeneous setting. We keep the assumption **(A)** as standing assumptions for the rest of the paper. Define

$$Y_t = C_1 t^{-1} + C_2 \varphi_t^{-2}(X_{T-t}^T) + C_3 \max\{k_2, k_3\} + (\alpha - 1)^{-1}(k_1 + \alpha k_3), \quad \alpha > 1, \quad (3.1)$$

where $C_1, C_2, C_3 > 0$ are constants which will be specified later. Let h_t be the solution of $\dot{h}_t = h_t Y_t$, $h_T = 1$. Recall that

$$q_t = \left(\frac{|\nabla^t u|^2}{u} \right) (t, X_{T-t}^T).$$

By studying the following process,

$$S_{t,\alpha} := h_t(q_t - \alpha \Delta_t u_t) - \beta n u_t \dot{h}_t \equiv h_t(q_t - \alpha \Delta_t u_t - n \beta u_t Y_t), \quad (3.2)$$

where $\beta > 0$ will also be specified later, we obtain the main result in this section.

Theorem 3.1 (Local gradient inequality of Li-Yau type). *Assume **(A)** holds. Let u be a solution of the heat equation (1.2) on D , which is positive and continuous on \overline{D} . For any $\alpha \in (1, +\infty)$, we have*

$$\frac{|\nabla^t u|^2}{u^2} - \alpha \frac{\Delta_t u}{u} \leq n \alpha^2 \left(\frac{2}{t} + \frac{2C_{\varphi,\alpha,n}}{\varphi_t^2(x)} + \max\{k_2, k_3\} + (\alpha - 1)^{-1}(k_1 + \alpha k_3) \right), \quad (3.3)$$

where $C_{\varphi,\alpha,n} := \sup_D \{(3 + \alpha^4(\alpha - 1)^{-1}n)|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\}$.

Remark 3.2. (1) When the manifold M is compact, consider the system (1.1)-(1.2). Under assumptions of Theorem 1.1, then $k_1 = k_3 = 0, k_2 = k$. Replacing Y_t in (3.1) by $C_1 t^{-1} + C_2 k$ and $S_{\alpha,t}$ in (3.2) by

$$h_t(q_t - \Delta_t u_t) - n \beta u_t \dot{h}_t,$$

we have the second assertion in Theorem 1.1 with a similar discussion as in the proof of Theorem 3.1.

(2) Adding the parameter β is to promise that α can be chosen for any $\alpha > 1$. This makes up the deficiency of traditional method applied in [3], where α only can be chosen from the interval $(1, 2)$.

Proof of Theorem 3.1. We first investigate the martingale property of $S_{t,\alpha}$. By (2.2) and (2.3), we find (modula differential of local martingales)

$$d(h_t q_t) \leq \left[-h_t u_t \|\text{Hess}_f^t\|_{HS}^2 + (\dot{h}_t + k_1 h_t) q_t \right] dt,$$

where $\text{Hess}_f^t(X, Y) = \langle \nabla_X^t \nabla^t f, Y \rangle$ for $X, Y \in TM$, $f = \log u$ and $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. It is easy to see that the process (X_{T-t}^T) is generated by $-\frac{1}{2}\Delta_t$. Observing (modula differential of local martingales)

$$\begin{aligned} d(\Delta_t u_t) &= [-u_t(\partial_t g_t, \text{Hess}_f^t)_{\wedge^2 T^*M} - u_t \partial_t g_t(\nabla^t f, \nabla^t f)] dt \\ &\geq [-u_t(\partial_t g_t, \text{Hess}_f^t)_{\wedge^2 T^*M} - k_3 u_t |\nabla^t f|^2] dt, \end{aligned}$$

where $(\cdot, \cdot)_{\wedge^2 T^*M}$ is the fiber metric on $\wedge^2 T^*M$ induced by g_t , we have

$$\begin{aligned} dS_{t,\alpha} &= dh_t q_t - n\beta u_t d(h_t Y_t) - n\beta d[u, \dot{h}]_t - \alpha dh_t \Delta_t u_t \\ &\leq [-h_t u_t \|\text{Hess}_f^t\|_{HS}^2 + \alpha h_t u_t (\partial_t g_t, \text{Hess}_f^t)_{\wedge^2 T^*M} - \alpha \dot{h}_t (\Delta_t u_t)] dt \\ &\quad + (\dot{h}_t + k_1 h_t + \alpha k_3 h_t) q_t dt - n\beta u_t h_t Y_t^2 dt \\ &\quad + n\beta u_t h_t [C_1 t^{-2} + C_2 c_\varphi(t, X_{T-t}^T) \varphi_t^{-4}(X_{T-t}^T)] dt - nC_2 \beta h_t d[u, \varphi_t^{-2}(X_{T-}^T)]_t, \end{aligned} \quad (3.4)$$

where $c_\varphi(t, x) = [3|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t](x)$. Observing $\Delta_t f_t = \frac{1}{u_t}(\Delta_t u_t - q_t)$, we have

$$\begin{aligned} dS_{t,\alpha} &\leq [-h_t u_t \|\text{Hess}_f^t\|_{HS}^2 + \alpha h_t u_t (\partial_t g_t, \text{Hess}_f^t)_{\wedge^2 T^*M} - \alpha \dot{h}_t u_t (\Delta_t f_t)] dt \\ &\quad + (k_1 + \alpha k_3) h_t q_t dt - (\alpha - 1) \dot{h}_t q_t dt - n\beta u_t h_t Y_t^2 dt \\ &\quad + n\beta u_t h_t [C_1 t^{-2} + C_2 c_\varphi(t, X_{T-t}^T) \varphi_t^{-4}(X_{T-t}^T)] dt - nC_2 \beta h_t d[u, \varphi_t^{-2}(X_{T-}^T)]_t. \end{aligned} \quad (3.5)$$

Since for any $a, b > 0$ such that $a + b = \alpha^{-1}$,

$$\begin{aligned} &\|\text{Hess}_f^t\|_{HS}^2 - \alpha (\partial_t g_t, \text{Hess}_f^t)_{\wedge^2 T^*M} = \{[a\alpha + b\alpha] \|\text{Hess}_f^t\|_{HS}^2 - \alpha (\partial_t g_t, \text{Hess}_f^t)_{\wedge^2 T^*M}\} \\ &= \left[a\alpha \|\text{Hess}_f^t\|_{HS}^2 + \alpha \left\| \sqrt{b} \text{Hess}_f^t - \frac{1}{2\sqrt{b}} \partial_t g_t \right\|_{HS}^2 - \frac{\alpha}{4b} \|\partial_t g_t\|_{HS}^2 \right] \\ &\geq \left[a\alpha \|\text{Hess}_f^t\|_{HS}^2 - \frac{\alpha}{4b} \|\partial_t g_t\|_{HS}^2 \right] \geq \frac{a\alpha}{n} (\Delta_t f_t)^2 - \frac{\alpha n}{4b} \max\{k_2^2, k_3^2\} \end{aligned} \quad (3.6)$$

at $(x, t) \in D$. In addition, with a similar calculation as in the proof of [3, Lemma 4.1], we have

$$\begin{aligned} -n\beta C_2 h_t d[u, \varphi_t^{-2}(X_{T-}^T)]_t &= 2n\beta C_2 h_t \varphi_t^{-3} \langle \nabla^t u_t, \nabla^t \varphi_t \rangle dt \\ &\leq 2nC_2 h_t [\varphi_t^{-1}((\alpha - 1)^{-1} n u_t)^{-1/2} |\nabla^t u_t| \times \beta((\alpha - 1)^{-1} n u_t)^{1/2} \varphi_t^{-2} |\nabla^t \varphi_t|] dt \\ &\leq [C_2 h_t \varphi_t^{-2} (\alpha - 1) u_t^{-1} |\nabla^t u_t|^2 + n^2 C_2 h_t \beta^2 (\alpha - 1)^{-1} u_t \varphi_t^{-4} |\nabla^t \varphi_t|^2] dt \\ &\leq \left\{ [(\alpha - 1) \dot{h}_t - (k_1 + \alpha k_3) h_t] q_t + C_2 (\alpha - 1)^{-1} \beta^2 n^2 u_t h_t \varphi_t^{-4} |\nabla^t \varphi_t|^2 \right\} dt. \end{aligned} \quad (3.7)$$

As $\frac{a}{n}(\Delta_t f_t)^2 + Y_t \Delta_t f_t \geq -\frac{nY_t^2}{4a}$, and combining this with (3.5), (3.6) and (3.7), we arrive at

$$\begin{aligned} dS_{t,\alpha} &\leq \left[-nh_t Y_t^2 u_t \left(\beta - \frac{\alpha}{4a} \right) + \frac{n\alpha}{4b} u_t h_t \max\{k_2^2, k_3^2\} \right] dt \\ &\quad + nu_t h_t (\beta C_1 t^{-2} + \beta C_2 C_{\varphi, \alpha, \beta, n} \varphi_t^{-4}(X_{T-t}^T)) dt, \end{aligned} \quad (3.8)$$

where $a + b = \alpha^{-1}$ and $C_{\varphi, \alpha, \beta, n} = \sup_D \{(3 + \beta^2(\alpha - 1)^{-1}n)|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\}$. Let

$$\left(\beta - \frac{\alpha}{4a}\right) C_1^2 \geq \beta C_1, \quad \left(\beta - \frac{\alpha}{4a}\right) C_2^2 - \beta C_2 C_{\varphi, \alpha, n} \geq 0, \quad \left(\beta - \frac{\alpha}{4a}\right) C_3^2 - \frac{\alpha}{4b} \geq 0.$$

Using

$$Y_t^2 \geq C_1^2 t^{-2} + C_2^2 \varphi_t^{-4}(X_{T-t}^T) + C_3^2 \max\{k_2^2, k_3^2\},$$

then the right side of (3.8) is nonnegative.

Consider the process $\{X_{T-s}^T\}_{s \in [0, t]}$ with $X_{T-t}^T = x$ and $(t, x) \in D$. Let $\tau(x) := \sup\{s < t : X_{T-s}^T \notin D, s \geq 0\}$ and $\sup \emptyset = 0$. Since $q_t - \alpha \Delta_t u_t - n u_t Y_t$ converges to $-\infty$ as t tends to $0 \vee \tau(x)$. Since $S_{t, \alpha}$ has nonpositive drift, we obtain $S_{t, \alpha}(x) \leq 0$ for $t \in (0, T]$, which implies

$$\frac{\frac{|\nabla^t u|^2}{u} - \alpha \Delta_t u}{u} \leq n\beta \left(\frac{C_1}{t} + \frac{C_2}{\varphi_t^2(x)} + C_3 \max\{k_2, k_3\} + (\alpha - 1)^{-1}(k_1 + \alpha k_3) \right), \quad (3.9)$$

where

$$\beta C_1 \geq \frac{\beta^2}{\beta - \frac{\alpha}{4a}}, \quad C_3 \beta \geq \frac{\beta^2 C_{\varphi, \alpha, \beta, n}}{\beta - \frac{\alpha}{4a}} \quad \text{and} \quad \beta^2 C_3^2 \geq \frac{\beta^2 \alpha}{4b(\beta - \frac{\alpha}{4a})}.$$

Choosing $\beta = \frac{\alpha}{2a} > 0$ such that $\frac{\beta^2}{\beta - \frac{\alpha}{4a}}$ is minimized, i.e. $\frac{\beta^2}{\beta - \frac{\alpha}{4a}} = \frac{\alpha}{a}$. Then,

$$\beta C_1 \geq \frac{\alpha}{a}, \quad C_2 \beta \geq C_{\varphi, \alpha, \beta, n} \frac{\alpha}{a}, \quad C_3^2 \beta^2 \geq \frac{\alpha^2}{4ab}.$$

Set $a = b = \frac{1}{2\alpha}$, $C_{\varphi, \alpha, n} := \sup_D \{(3 + \alpha^4(\alpha - 1)^{-1}n)|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t + 2\partial_t)\varphi_t\}$. Then $\beta = \alpha^2$. Selecting C_1, C_2, C_3 such that

$$\beta C_3 = \alpha^2, \quad C_2 \beta = 2C_{\varphi, \alpha, n} \alpha^2, \quad \beta C_1 = 2\alpha^2,$$

we complete the proof by taking these constants C_1, C_2, C_3 into (3.9). \square

Now, we turn to studying the Li-Yau type inequality with lower order term. Our discussion is still based on assumption **(A)**. Let

$$\tilde{S}_t = \frac{|\nabla^t u_t|^2}{u_t} - \Delta_t u_t - n u_t \tilde{Z}_t,$$

where $\tilde{Z}_t = C_1 t^{-1} + C_2 \varphi_t^{-2}(X_{T-t}^T) + C_3$ with constants $C_1, C_2, C_3 > 0$ to be specified later. Let

$$c_\varphi := \sup_D \{3|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\}.$$

By a similar calculation as in the proof of Theorem 3.1, we have

$$\begin{aligned} d\tilde{S}_t \leq & \left[-\frac{a}{n u_t} \left(\frac{|\nabla^t u_t|^2}{u_t} - \Delta_t u_t \right)^2 + (k_1 + k_3) \frac{|\nabla^t u_t|^2}{u_t} + \frac{n}{4b} u_t \max\{k_2^2, k_3^2\} \right] dt \\ & + n u_t (C_1 t^{-2} + C_2 c_\varphi \varphi_t^{-4}(X_{T-t}^T)) dt - n C_2 d[\varphi^{-2}(X_{T-}^T), u]_t, \end{aligned}$$

where $a + b = 1$. Since

$$\begin{aligned} -nC_2 d[\varphi_t^{-2}(X_{T-t}^T), u]_t &= 2nC_2 \varphi_t^{-4}(X_{T-t}^T) \left\langle \frac{\nabla^t u_t}{u_t}, \varphi_t \nabla^t \varphi_t(X_{T-t}^T) \right\rangle u_t dt \\ &\leq 2nC_2 u_t \left\| \frac{|\nabla \cdot u|}{u} \right\|_D \|\varphi \cdot \nabla \cdot \varphi\|_D \varphi_t^{-4}(X_{T-t}^T) dt, \end{aligned}$$

we arrive at

$$\begin{aligned} d\tilde{S}_t &\leq \left[-\frac{a}{nu_t} \left(\frac{|\nabla^t u_t|^2}{u_t} - \Delta_t u_t \right)^2 + nu_t \left(\frac{C_1}{t^2} + \frac{C_2}{\varphi_t^4(X_{T-t}^T)} \right) \right. \\ &\quad \times \left(c_\varphi + 2 \left\| \frac{|\nabla \cdot u|}{u} \right\|_D \|\varphi \cdot \nabla \cdot \varphi\|_D \right) \Big] dt + nu_t \frac{k_1 + k_2}{n} \left\| \frac{|\nabla \cdot u|}{u} \right\|_D^2 dt \\ &\quad + \frac{1}{4b} nu_t \max\{k_2^2, k_3^2\} dt. \end{aligned}$$

Then, letting $C_1 = \frac{1}{a}$, $C_2 = \frac{1}{a} \left[c_\varphi + 2 \left\| \frac{|\nabla \cdot u|}{u} \right\|_D \|\varphi \cdot \nabla \cdot \varphi\|_D \right]$, and

$$C_3 = \sqrt{\frac{k_1 + k_2}{an}} \left\| \frac{|\nabla \cdot u|}{u} \right\|_D + \sqrt{\frac{1}{4ab}} \max\{k_2, k_3\},$$

we obtain

$$d\tilde{S}_t \leq -\frac{\tilde{S}_t}{nu_t} \left(\frac{|\nabla^t u_t|^2}{u_t} - \Delta_t u_t + nu_t Z_t \right) dt,$$

and on $\{\tilde{S}_t \geq 0\}$, the process \tilde{S}_t has nonpositive drift. Set $a = b = \frac{1}{2}$. Consider $\{X_{T-s}^T\}_{s \in [0, t]}$ starting from x at $s = t$. Since \tilde{S}_t goes to $-\infty$ as $s \rightarrow 0 \wedge \tau(x)$, where $\tau(x) = \sup\{s < t : X_{T-s}^T \notin D, s \geq 0\}$, we obtain the following result.

Theorem 3.3 (Local Li-Yau type with lower order term). *Assume (A) holds. Let $u = u(x, t)$ be a positive solution of (1.2) on D . Then*

$$\frac{|\nabla^t u|^2}{u^2} - \frac{\Delta_t u}{u} \leq \frac{2n}{t} + \frac{2nc_\varphi}{\varphi_t^2(x)} + \max\{k_2, k_3\}n + \left(4n \frac{\|\varphi \cdot \nabla \cdot \varphi\|_D}{\varphi_t^2(x)} + \sqrt{2n(k_1 + k_3)} \right) \left\| \frac{|\nabla \cdot u|}{u} \right\|_D, \quad (3.10)$$

where $c_\varphi := \sup_D \{3|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\}$.

When $D = B_{\beta, T}$, φ is chosen in (2.8). Moreover, let $\text{Cut}_t(x_0)$ be the set of the g_t cut-locus of x_0 on M . Since the time spent by X_{T-t}^T on $\bigcup_{t \in [0, T]} \text{Cut}_t(x_0)$ is a.s. zero (see [13]), the differential of the brackets $[\varphi \cdot (X_{T-t}^T), u]_t$ may be taken as 0 at points where φ_t^{-2} is not differentiable. Therefore, Theorem 3.1 and Theorem 3.10 hold for this φ , and we have the following estimates following from (2.10) and (2.11).

$$c_\varphi = \sup_{B_{\rho, T}} \{3|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\} \leq \frac{\pi^2(n+3)}{4\rho^2} + \frac{\pi}{2}(k_1 + k_2); \quad (3.11)$$

$$\begin{aligned} C_{\varphi, \alpha, n} &= \sup_{B_{\rho, T}} \{(3 + \alpha^4(\alpha - 1)^{-1}n)|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t\} \\ &\leq \frac{\pi^2(n+3 + \alpha^4(\alpha - 1)^{-1}n)}{4\rho^2} + \frac{\pi}{2}(k_1 + k_2). \end{aligned} \quad (3.12)$$

Under assumption of Theorem 1.2, $k_1 = 0, k_2 = k_3 = k$, and by the estimates (3.11) and (3.12), we prove the rest result in Theorem 1.2 from (3.3) and (3.10) directly.

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References

1. Arnaudon, M., Coulibaly, K., Thalmaier, A., Brownian motion with respect to a metric depending on time: definition, existence and applications to Ricci flow, C. R. Math. Acad. Sci. Paris, 346 (2008), 773–778.
2. Arnaudon, M., Driver, B. K., Thalmaier, A., Gradient estimates for positive harmonic functions by stochastic analysis, Stochastic Process. Appl., 117 (2007), 202–220.
3. Arnaudon, M., Thalmaier, A., Li-Yau type gradient estimates and Harnack inequalities by stochastic analysis, Probabilistic approach to geometry, 29–48, Adv. Stud. Pure Math., 57, Math. Soc. Japan, Tokyo, 2010.
4. Bailesteanu, M., Cao, X., Pulemotov, A., Gradient estimates for the heat equation under the Ricci flow, J. Funct. Anal., 258 (2010), 3517–3542.
5. Cao, X., Differential Harnack estimates for backward heat equations with potentials under the Ricci flow, J. Funct. Anal., 255 (2008), 1024–1038.
6. Cao, X., Hamilton, R. S., Differential Harnack estimates for time-dependent heat equations with potentials, Geom. Funct. Anal., 19 (2009), 989–1000.
7. Cheng, L. J., Reflecting Diffusion Process on Time-Inhomogeneous Manifolds with Boundary, preprint, 2012.
8. Guenther, C. M., The fundamental solution on manifolds with time-dependent metrics, J. Geom. Anal., 12 (2002), 425–436.
9. Hamilton, R. S., The formation of singularities in the Ricci flow, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.
10. Ni, L., Ricci flow and nonnegativity of sectional curvature, Math. Res. Lett., 11 (2004), 883–904.
11. Yau, S. T., Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math., 28 (1975), 201–228.
12. Zhang, Q. S., Some gradient estimates for the heat equation on domains and for an equation by Perelman, Int. Math. Res. Not., 2006, Art. ID 92314, 39 pp.

13. Kuwada, K., Philipowski, R., Non-explosion of diffusion processes on manifolds with time-dependent metric, Math. Z., 268(2011), 979–991.
14. Liu, S., Gradient estimates for solutions of the heat equation under Ricci flow, Pacific J. Math., 243 (2009), 165–180.
15. Shen, Y., On Ricci deformation of a Riemannian metric on manifold with boundary, Pacific J. Math., **173**(1996), 203–221.
16. Sun, J., Gradient estimates for positive solutions of the heat equation under geometric flow, Pacific J. Math., 253 (2011), 489–510.

4 Appendix

Proof of Theorem 2.6. Assume that $0 < u \leq e^{-3}$ (the assumption will be removed in Theorem 2.6 through replacing u by $e^{-3}u/\|u\|_D$). Using the fact that (modula differential of local martingales)

$$\nabla^t(u_t(1 - \log u_t)^2) = (\log^2 u_t - 1)\nabla^t u_t \quad \text{and} \quad d(u_t(1 - \log u_t)^2) = -q_t \log u_t dt,$$

we get

$$\begin{aligned} dS_t &= \frac{1}{2}d\left\{\frac{|\nabla^t u_t|^2}{u_t}\right\} - u_t(1 - \log u_t)^2 dZ_t - Z_t d(u_t(1 - \log u_t)^2) \\ &\quad - 2C_2(1 - \log u_t)(1 + \log u_t)\varphi_t^{-3}(X_{T-t}^T) \langle \nabla^t u_t, \nabla^t \varphi_t(X_{T-t}^T) \rangle dt \\ &\leq \frac{1}{2}k_1 q_t dt + u_t(1 - \log u_t)^2 [C_1 t^{-2} + C_2 c_\varphi(t, X_{T-t}^T)\varphi_t^{-4}(X_{T-t}^T)] dt + \log u_t q_t Z_t dt \\ &\quad - 2C_2(1 - \log u_t)(1 + \log u_t)\varphi_t^{-3}(X_{T-t}^T) \langle \nabla^t u_t, \nabla^t \varphi_t(X_{T-t}^T) \rangle dt, \end{aligned}$$

where $c_\varphi(t, x) = [3|\nabla^t \varphi_t|^2 - \varphi_t(\Delta_t - 2\partial_t)\varphi_t](x)$. Now from $u \leq e^{-3}$, we get $2\log u_t \leq -3(1 - \log u_t)/2$, this together with $|1 + \log u_t| \leq 1 - \log u_t$ yields

$$\begin{aligned} dS_t &\leq (\log u_t - 1) \left\{ \left[\frac{1}{2} \left(\frac{3}{2}Z_t - k_1 \right) q_t - u_t(1 - \log u_t)(C_1 t^{-2} - C_2 c_\varphi(t, X_{T-t}^T)\varphi_t^{-4}(X_{T-t}^T)) \right] dt \right. \\ &\quad \left. - C_2(1 - \log u_t)2\sqrt{2}\varphi_t^{-2}(X_{T-t}^T)|\nabla^t \varphi_t(X_{T-t}^T)|\sqrt{u_t}\varphi_t^{-1}(X_{T-t}^T)\sqrt{\frac{q_t}{2}}dt \right\} \\ &\leq (\log u_t - 1) \left[(Z_t - k_1)\frac{1}{2}q_t - u_t(1 - \log u_t)^2 \right. \\ &\quad \left. \times (C_1 t^{-2} + C_2 (c_\varphi + 4|\nabla^t \varphi_t(X_{T-t}^T)|^2)\varphi_t^{-4}(X_{T-t}^T)) \right] dt. \end{aligned}$$

Let $C_1 = 1, C_2 = \sup_D \{c_\varphi + 4|\nabla^t \varphi_t|^2\}$, and $C_3 = k_1$, we get

$$dS_t \leq -(1 - \log u_t)(Z_t - k_1) \left[\frac{1}{2}q_t - u_t(1 - \log u_t)^2 Z_t \right] dt.$$

Now we consider the process $\{X_{T-s}^T\}_{s \in [0, t]}$ starting from x at time t . This process has nonpositive drift on $\{S_s \geq 0\}$. On the other hand, S_s converges to $-\infty$ as $s \rightarrow 0 \vee \tau(x)$, where $\tau(x) = \inf\{s \leq t : X_{T-s}^T \notin D, s \geq 0\}$. \square